SENSE AND OBJECTIVITY IN FREGE'S LOGIC

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The essentials of Frege's revolutionary logic appeared in his *Begriffsschrift* (B, 1879). Important aspects of its philosophical basis, and its significance for the foundations of mathematics, appeared in *The Foundations of Mathematics* (FA, 1884). Six years later, at the beginning of the 1890s, Frege published three articles that mark significant changes in his conception: "Function and Concept" (FC, 1891), "On Sense and Reference" (SR, 1892) and "Concept and Object" (1892). Notable among these changes are: (a) The systematic distinction between the sense and the reference of expressions as two separate ingredients of their meaning. (b) The extension and generalization of the notion of function to include the conception of concepts and relations as functions to truth-values, and the corresponding conception of the two truth-values as objects.[1] These changes were immediately incorporated in the mature, authoritative exposition of his logic in his *magnum opus: Basic Laws of Arithmetic* (BL), whose first volume appeared in 1893.

What is the role of the notion of sense and of the distinction between sense and reference in Frege's logic? Is there a systematic connection between the two points (a) and (b) mentioned above, so that their being incorporated together in Frege's mature logic is not accidental? These questions, I believe, are central to understanding Frege's mature conception.

In the sequel, after presenting the problem in a sharper way (A), I shall sketch what seems to me the general direction of an answer (B-C), and then add further clarifications of related issues (E-D). In three sentences the general direction is this: Logic, in its wide sense, is, for Frege, the science of justification and objectivity. These are correlative notions: the objective is what is justifiable, and justification requires objective standards. The role of the notion of sense in this enterprise is in establishing the objectivity of the basic truths of a domain (including logic itself), which is accomplished by presenting these truths as expressing features of the ways in which the objects of the domain are given to us. This appeal to objects and their modes of presentation gives a particular realistic turn to Frege's

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1 The extension of the notion of a function was already a major theme in B (see B section 9). But there it was introduced as an expression, and was perhaps flawed by a confusion of sign and thing signified, or expression and content expressed. Moreover, since Frege thought there in terms of a general undifferentiated notion of content, and he did not clearly distinguish a function from its values, essential features of his conception remained unclear. I shall not go into the details here.

notion of objectivity: One face of it connects it to justification; another, to objects and their modes of being given to us.[2]

(A) Two Characterizations of Sense

1. The Core Idea - Sense as a Mode of Being Given

Frege's notion of sense is usually presented as stemming from epistemological considerations, as carrying the "cognitive value" or informativeness of expressions and sentences. Various formulations of Frege's provide support for such a view, notably the famous paragraph at the beginning of SR, where Frege argues for the need to distinguish the sense from the reference of expressions. Apart from some general principles that govern these notions, the reference of a term is explained as what it denotes in its use in simple sentences, and what these sentences are about.[3] The sense of a term is introduced, on this conception, as the mode or way in which its reference is given to us. Thus "The morning star is the evening star" is true in that the two names have the same reference, and it is informative in that they have or express different senses in which their "cognitive value" is contained; the sentence as a whole expresses the thought to the effect that these two senses belong to the same reference.[4]

As I said before, this is a prevalent conception of Frege's notion of sense, which finds its clearest formulation in Frege's own writings primarily in SR. Following the terminology I used in my book (p. 7) I call this notion of sense "the core idea" of sense. There is something "local" and lexical about this notion of sense: One begins with the senses of individual simple names, and moves on "from the bottom up" to more complicated ones. It is quite late in the article (32/62) that thoughts are presented as the senses of complete sentences (where presumably the notion of sense is taken as already understood). And nowhere in this article

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[2] Sense, in Frege's philosophy, appears to belong to logic most clearly in definitions and in his theory of definition. I shall not go into these issues here. In its strict sense a definition for Frege is a stipulation of synonymy, as it stipulates identity of sense between definiendum and definiens. The definiendum in fact gets its meaning from the definiens through this stipulation. Clearly in such a conception the distinction between reference and some notion of sense is mandatory. Frege also recognized another, less strict notion of definition in which the definiendum is a term in current use, already endowed with meaning. Here again questions of adequacy and efficiency require appeal to some notion of sense. What the roles of these kinds of definition in Frege's logic are, whether these notions are the same, and whether they are the same as the ones explicated in the text are serious problems I shall not go into here.

[3] In these governing principles lies the great novelty of Frege's notion of reference; I shall not expand on it here. On the significance of this appeal to the notion of aboutness here, and for references, see chapter 7 of my book (1996).

[4] Important ingredients of the core idea already appear earlier, though without the systematic terminology. See, for instance B, section 8; FA, section 62. Compare also my book (1996) pp. 44, 54, 111.

does there occur the crucial idea (central in other writings) that the sense of an expression is a constituent of a thought - the particular contribution the expression makes to expressing this thought.[5]

Being epistemically governed, this notion of sense is also individuated in epistemic terms. Frege often proposes or assumes that two senses are the same if and only if whenever one knows them one knows they are the same. Put in different terms, the criterion says that two expressions express the same sense if and only if it is impossible to understand them both, yet fail to know that they express the same sense.[6]

This account may be correct as far as it goes, but it does not go far enough. For Frege introduces and uses the notion of sense in his distinctly logical writings (e.g. FC, BL). One can naturally wonder about the role this notion of sense plays there, in Frege's logical doctrines, even when one grants the above account of its role in Frege's epistemology, and in his account of various features (such as cognitive value) of natural language sentences and expressions. Logic, as it is often conceived, is concerned with a clear and systematic presentation of deductions and proofs, and so it was generally conceived by Frege as well.[7] Sense, it may be claimed, does not belong here, even granted its importance in accounting for other, non-logical, features of sentences (in a natural language as well as a logical one).

To highlight the point it is quite typical that, although modern systems of logic derive their essentials from Frege's logic (the generalized function-argument conception, the conception of sentential logic as the logic of "truth functions", i.e. functions over truth-values, the basics of quantification theory, etc.), the most distinguished exception is Frege's notion of sense, which is hardly mentioned in most courses and texts of classical modern logic. Logicians do not seem to need this notion; many of them hardly know anything about it, and many of those who do, who tend to be more philosophically oriented, explicitly and

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5 This article is unique among Frege's writings in being directly concerned solely with natural language. This is important because, on the one hand, it clearly shows that Frege intended to apply his notions of sense and reference to natural language sentences and expressions, while on the other, it may explain the particular way in which the introduction of the notion of sense here diverges from the way it is presented in later writings of a more logical orientation.

6 See, for instance, "A Brief Survey...", PW p. 197. The principle is also assumed (though not stated) by Frege in T (65/25; cf. my book p. 70). It should be noted that there are other criteria, which Frege proposes and uses for sameness of senses and thoughts. See, for instance PW 140, letter to Husserl of 9 Dec. 1906, PMC 70. For a broader discussion cf. my book pp. 214-217.

7 See for instance "Logic", PW pp. 3, 4; "17 Key Sentences...", PW p. 175. This conception of logic is related to Frege's repeated claim that truth, as distinct from the recognition of truth, is the subject-matter of logic (PW 128/139; the beginning of T)

doctrinairely reject it. All this may strengthen the suspicion that, in trying to incorporate the notion of sense into his logic, Frege was appealing to a different notion of sense.[8]

2. Sense as a Constituent of Thought

There is, indeed, another conception, or perhaps merely a different emphasis in the conception of sense, which is found mainly in Frege's later writings, and is dominant in his logical works. In this latter conception, thought is the primary notion, where senses are conceived as parts of thoughts (their "building blocks"). The sense of a (declarative) sentence is identified with the thought it expresses and the senses of its constituent expressions are presented as their contribution to that thought. This is the dominant conception of sense in Frege's later writings. It occurs as early as FC (13-14/29), but gets its conspicuous expression in the celebrated section 32 of BL:

> The names, whether simple or themselves composite, of which the name of a truth-value consists, contribute to the expression of a thought, and this contribution of the individual [component] is its sense.

Again, following the terminology I used in my book (p. 7-8), I shall refer to it as the thought-constituent notion of sense. This conception is obviously connected with the context-principle, and seems to be significantly different from the "local-lexical" conception of the "core idea". The difference is perhaps most conspicuous if one considers a thought (as many believe that Frege did) as a Platonic entity, existing in itself, independently of human minds (though perhaps, as Frege says in FA, not of The Mind), and being true or false independently of our cognitive limitations. This notion of thought has not so much to do with knowledge and modes of presentation, as with logic and logical relations: A thought is what is true or false, and what stands in logical relations of deducibility, contradiction and so forth, with other thoughts. Unlike the core idea, senses of sub-sentential expressions are not conceived of as modes of presentation of their references, carrying their "cognitive value", but as constituents of thoughts, constituents whose very being and individuation depends entirely on that of the thoughts containing them and their logical structure.

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8 With all its reliance on Frege's ideas, there are some respects in which modern logic took a course different from his, notably in its reliance on set theory and model theory. It may therefore appear that the role played by the notion of sense in Frege's logic is somewhat analogical to the role of the notion of a model in post-Tarskian logic. In some extensions of standard model theory in modal logic this idea has found explicit formulations, e.g. by Hintikka, who has explicitly suggested explicating Frege's notion of sense in terms of the extension of terms in various possible worlds. See e.g. his 1969, p. 105. I believe that this proposal is mistaken and misses some essential features of Frege's notion of sense (e.g. its particular perspectival cognitive character, and its intentionalistic nature (see my nook 1996, ch. 1), but it may support the general point made above, that in Frege's logic, the notion of sense played a similar role to that played by model theoretic notions in later developments of logic.

It is, moreover, difficult to see how sense can be conceived as mode of presentation when the notion of thought is regarded as the primary notion of sense and thus the basis of any other kind of sense. For it is difficult to see what the notion of the mode of presentation of a truth value could amount to, even apart from the grave difficulties in conceiving truth values, regarded as the references of sentences, as objects. As I argued in my book, Frege's notion of sense is primarily intentionalistic, and the idea of mode of presentation is vital to it. This idea is intuitively well understood (or at least relatively so) with regard to objects - primarily concrete "ordinary" objects, and with some refinements and adjustments, abstract objects; it begins, however, to creak and squeak when applied to concepts and functions; it seems completely ad hoc and unintelligible with regard to truth-values. It therefore appears that we are faced here with two unequivalent characterizations of sense, and one may wonder whether the primacy of thoughts and the conception of sense as thought-constituent can be naturally reconciled with the core idea of sense.

Looking back at our opening questions about the role of the notion of sense in Frege's logic, it appears now that the notion of sense relevant to logic is not the epistemic notion of the core idea (a mode of being given), but the later idea of a thought-constituent.[9] Logic, as mentioned before, is concerned with a clear and systematic exposition of inferences or in justifying and establishing truths on the basis of other truths. Whatever is relevant to this task is logic's concern. A thought is what is true or false, and in that sense thoughts form the subject matter of logic. Moreover, it is logic (including the study of the structures and properties of logical languages) that determines what a thought is: A thought is that part of the content of a sentence, which is required for a clear and systematic exposition of the logical relations it can enter into, and, in particular, what is required for establishing and justifying the truth of a statement on the basis of other truths.[10]

Frege's repeated objection to the conception of logic as an abstract, purely formal calculus that can be interpreted in various models is a facet of his insistence on conceiving logical formulae as "full blooded" statements, expressing thoughts which are true or false.[11] This is an essential point in his conception of logic, and it is a point that places the notion of sense immediately into the very center of logic. Logic for Frege is concerned primarily with thoughts and it is what determines the parameters that make a thought what it is.[12]

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[9] See the proviso about definitions in note 1.

[10] This is central to both the "Logic" of the 1880s and to the later "Logic" of the 1890s (as well as many other writings), both written probably as parts of a general textbook on logic. See PW pp. 1-8; 126-151. The notion of thought here stems from “conceptual content” of B, which is characterized in similar terms (see, B section 3).

[11] See my article "Frege's Early Conception of Logic", Epistemologia VIII (1985), pp. 125-40.

[12] See Frege's PW (1979) pp. 197-8; compare also C. Diamond, The Realistic Spirit (1991), pp. 115-144, especially 117-120.

These points about the difference between the core-idea and the thought-constituent conceptions of sense may appear to threaten the coherence of Frege's notion of sense. It is not clear, however, that they amount to more than a difference in emphasis. I have mentioned before that the difference is most conspicuous on a Platonic conception of thoughts (and senses). It may thus appear that the difference may be diminished if this Platonic conception is rejected, as it is arguable that it suggests a misleading picture of Frege's conception of thought, and that a thought, as well as the conception of the structure of a thought and the contribution its constituents make to it, are themselves epistemic notions, or epistemically constrained. This is the direction taken in my book, where the connection between these two faces of sense is explained along these lines (see, e.g. p. 15). Yet recovering the coherence of Frege's notion of sense in this way may be gained at the price of rendering it in its entirety irrelevant to logic. For it is doubtful whether such an epistemic notion can carry the burden of an objective conception of thought as what is true or false (under bivalence) and as what stands in logical relations, in the strong classical sense. It is also for this reason that the difference seems significant enough to deserve further attention, and that the relations between the two notions (or two aspects of the notion of sense) require careful study.

(B) Sense and Justification - The Coherence of Frege's Notion of Sense

The previous considerations seem to threaten to tear Frege's notion of sense apart, and to show that the notion of sense relevant to logic is the thought-constituent notion. This, I believe, is too hasty a conclusion. I shall try to show this by pointing out the role of the core idea in logic and in establishing the objectivity of logic. Being thus placed at the center of logic, the core idea is seen as correlative and complimentary to the thought-constituent conception of sense. With reference to our opening question about the role of the notion of sense in Frege's logic, we shall thus see that it is precisely its role in logic, conceived as the science of objectivity, which restores the coherence of the Fregean notion of sense.

The key to understanding this is the role of the notion of sense in establishing the objectivity of a domain of thoughts. The criterial sign of objectivity, for Frege, is justification or justifiability: something is objective insofar as it is justifiable or as statements about it are. The first of the three principles in the Introduction to FA is the demand to distinguish between the logical and the psychological, the objective and the subjective. These are parallel distinctions. It appears therefore that the objective is the logical. And logic, as Frege makes clear on numerous occasions, has to do with the justification of propositions. (See the Preface to B; FA section 3; PW p. 3; PW 147.) The main tenor of FA is to establish the objectivity (or "objective factuality", as Frege sometimes says) of arithmetic by clarifying the grounds or justification of arithmetical propositions. Logic, objectivity and justification form an inseparable triad for Frege. This, however, raises a question, for in its simple sense (and so also in Frege) what is objective is what is "out there" in the world, or what concerns objects that are out there in the world, and which are accessible and can be examined by different people, from different perspectives, etc. What, then, is the relationship between these two aspects of the objective: the logical aspect, on the one hand, and being concerned with and based on objects in the world, on the other? In trying to answer this, I believe, we must appeal to the third element of the triad - the notion of justification. What is objective is only what is justifiable or what is used in a justification; in short, what is in the justification-space. Logic is not only itself objective in this sense, but is constitutive of objectivity. It is what sets the standards for justification and objectivity, what constitutes the justification-space.

The main task Frege set up for himself was to establish the objectivity of various domains - particularly mathematics (arithmetic). For him this means presenting statements in this domain as justified or at least justifiable. Logic was the paradigm and primary means of such justification. Deductions and proofs are chains of justifications of some truths on the basis of others. In this sense, logic, being the paradigm of justification is also the heart of objectivity. It is not only a paradigm of objectivity, but it is what sets the standards of objectivity, and is thus constitutive of the very notion of objectivity. It is obvious that on this conception both logic and objectivity have to do mainly with the notion of sense as thought-constituent. The core-idea notion of sense seems to be out of the picture.

But, and this is the main point, there is another level, or stage of justification and objectivity, which is not strictly deductive, but can still be considered as logical in the wide sense. Logic in this wide sense is precisely the theory of justification or justifiability. [13] This other level is where we justify the basic truths of a domain not by proving them or deriving them from more fundamental truths (since there aren't any), but by showing them to be clear, justified, or evident by the way "their objects", the objects they are about, are given to us. One can even say that these basic truths are justified by the fact that they express (aspects of) the ways "their" objects are given us, or, in other words, by the modes of presentation or senses of these objects.

Frege's primary example of this was geometry. Geometrical truths (theorems) are objective in being justifiable. They are justified, through logical proofs, on the basis of other geometrical truths, and ultimately by the basic truths, or axioms of geometry. But what about these axioms themselves? Obviously, they cannot be derived logically from more basic truths. Should we say they are not justifiable, and therefore not objective? Certainly not. This would ruin the objectivity of the whole edifice built on the basis of these axioms(cf. FA section 26). It is here that this other form of justification is used. The axioms are justified on the basis of the ways their objects, the objects of geometry, are given to us: "Everything geometrical must

[13] See, for instance, "Logic", PW p. 3; "17 Key Sentences..." ibid. 175. Cf. also my book (1996) pp. 40-46.

be given originally in intuition" (FA, p. 75). The axioms, one may say, express at least some basic aspects of these modes of presentation of the geometrical objects.[14]

Frege sets himself the task of constructing something similar for arithmetic, thus establishing solid foundations for mathematics. The defects he found in arithmetic were not only that many arithmetical proofs were unclear or obviously faulty, but also that the whole science of arithmetic lacked an objective basis. To remedy the former fault it was necessary, according to Frege, to present arithmetical thoughts in a systematic logical language that would render their proofs transparent and detectable. That was obviously the task of logic, or of constructing a logical language. In principle this was accomplished by the logical language of B (1879). But the latter deficiency was no less severe, and called for no lesser a task: the task, namely, of presenting the objects of arithmetic, the objects the axioms are concerned with (namely, numbers), in such a way that the axioms themselves will be justified.

Frege's logicism - the program of presenting arithmetic as logic- was designed to solve both problems. Expressing arithmetical truths in the language of the *Begriffsschrift* enabled him to present their proofs completely and systematically. But it served a further and, in a way, more basic aim. Being convinced that there is no other way in which numbers can be construed that can justify their axioms, Frege thought that the only way to achieve this goal was to construe them as logical objects. This, together with a logical rendering of other arithmetical notions, would enable us to present and justify the arithmetical axioms as logical truths. Again, the crucial step here is the double move of regarding the axioms, the basic truths of arithmetic, as being about objects of a certain kind (numbers), and regarding the ways these objects are given us as justifying these axioms, thus establishing the objectivity of the whole edifice of arithmetic. This double move may explain Frege's persistent view that establishing the objectivity of arithmetic requires an explication of the nature of numbers and their way of being given to us. This was not just a sort of a Socratic wondering about essences; it was rather a requirement implied by the conception of objectivity under discussion, and by the task of establishing the objectivity of arithmetic. It is evident (as, e.g., the structure of FA makes manifest) that had Frege been satisfied with other ways of establishing the objectivity of arithmetic and explicating the nature of numbers, he would have seen no need and no point in reducing arithmetic to logic. But, of course, for most of his life Frege was convinced that no non-logical account was possible, and was thus "forced" into his logicistic project. Frege's logicism is, from this point of view, not only a reductivist program and a technical achievement, but also a philosophical discovery: it is the discovery that one can present the basic truths of arithmetic as being about objects whose mode of presentation to us as logical objects justifies these truths. Let me elaborate a bit.

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[14] Scholars have debated the question of whether this way of being given is basically Kantian spatial intuition, and whether Frege can be regarded as a Kantian in this respect. I shall not discuss this here. Cf. Dummett, IF (1981), pp. 463-470.

(C) Logical Objects - Frege's Fundamental Principle

The idea of logical objects is a notorious one. It was not part of the lore of logical tradition, which makes it even more puzzling why Frege held so firmly to it.[15] Moreover, the inconsistency of the fundamental axiom (V) in BL, which concerns logical objects, gave a fatal blow to Frege's "life-project", according to his own admission. Why then was he so persistent (some would say obsessed) about logical objects? I suggest that the answer is important for understanding not only Frege's conception of logic and mathematics, but also his conception of objectivity and sense, and their interrelationships. I offer a partial answer to this puzzle by suggesting two general points and a more specific one:

(a) The idea that any truth or any meaningful statement must be about something, primarily about objects.[16]

(b) The idea that the basic truths of a domain are objective in that they are justifiable, and that the justification of such truths or axioms can be attained in terms of the ways the objects they are about are given to us. (This is the main thesis argued for here).

There must of course be objects, in order for there to be ways in which they are given to us. Hence, these two principles imply that the objectivity of logic as a system of truths based on logical axioms rests on the existence of logical objects, whose ways of being given to us justify the axioms (even if only partially).

(c) Now the specific point: The specific reduction of arithmetic to logic suggested by Frege required appeal to classes (in one version or another), and the immediate Fregean question was how these classes are given to us. Since this is conceived as a reduction to logic, then, if the above position is valid, these classes must be given to us as logical objects. Giving up appeal to how these objects (classes) are given to us as logical objects (their senses) means giving up an account of the objectivity of arithmetic and parts of logic. This was a price Frege was unwilling to pay, and it is hard to blame him for this.

The last point involves some technical questions concerning the exact nature of the reduction involved and the set theory assumed. I shall not get into these issues here, but will confine myself to the essential general point: that on Frege's conception the objectivity of

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15 "Object-producing principles" were at the basis of Frege's philosophy as early as FA (1884). There Frege adopted a version of the "Hume-Principle", which says, roughly, that The-object-of-F is the same as The-object-of-G iff the concepts F and G are equinumerous (equipotent). George Boolos saw this as a principal reason for regarding Hume's principle as non-logical (see his, 1990, 261-77).

16 Frege held this doctrine throughout his career. See e.g., the early "Dialogue with Punjer on Existence" (PW 53-60), and the late "Numbers and Arithmetic", where he writes: "I, for my part, never had doubt that numerals must designate something in arithmetic, if such a discipline exists at all [...] We do after all make statements of number. In that case what are they used to make an assertion about?..." (PW 275). The *locus classicus* of this view is FA section 46, and ch. IV. For a detailed discussion of this point see ch. 7 of my book.

logic and its basic truths relies on there being logical objects and on the ways they are given to us. In order to see that, it is worthwhile to recall some features of the way in which logical objects are introduced (or discovered) in Frege's mature logical system. As early as 1891 (cf. FC, p. 9-10/26) Frege presented the basic idea of what was to become the fifth axiom of BL. He explains and defends a move from a general equivalence of the form $(x)(Fx\leftrightarrow Gx)$, what he calls there an "equality holding generally between values of functions", to an identity between objects - the "courses (or ranges) of values" of these concepts: x^Fx=x^Gx.

The move from the former to the latter - from the equivalence to the identity - is, according to Frege, irresistible and reliant on a fundamental and indemonstrable logical principle. It is this relationship that expresses the way the logical objects, the courses of values of functions (including extensions of concepts), are given to us. This then is the way in which we get to the logical objects - objects which are given to us by way of logic. Ignoring irrelevant subtleties, we may regard these objects, the courses of values, as classes. The above, then, is the move by which we get classes as logical objects with which the relevant part of logic - predicate logic - is concerned. They are given to us in a particular way which is expressed by that elementary move from general equivalence to identity. And it is this mode of being given that is purported to justify the relevant axiom (axiom V), which says roughly x^Fx = x^Gx $\leftrightarrow$ $(x)(Fx\leftrightarrow Gx)$, as the axiom expresses an essential aspect of this mode.

Of course, I do not intend to defend here an axiom that has been proved inconsistent. This particular account of the way the logical objects are given to us has been proved a failure.[17] But this does not mean that the basic position on the relation between objectivity and the ways objects are given to us - a position from which this account emerged - is faulty. It is this position, in its Fregean version, that I am trying to explain. Even if Frege's particular account was a failure, it is important to understand an account of what it was intended to be. It may be relevant to recall that even when, towards the end of his life, Frege gave up the logicistic approach to arithmetic, he still kept unquestioned the view argued for here: that accounting for the objectivity of arithmetic requires an explication of the nature of arithmetical objects and how they are given to us, which he then suggested should be construed on the basis of geometry (cf. "Number and Arithmetic" (PW 275-7), and "A New Attempt..." (PW 278-281)). Moreover, his qualms about axiom V of BL notwithstanding, he never questioned the very idea of logical objects and the need to appeal to them in accounting for the objectivity of logic. He kept talking of the "logical source of knowledge", which requires logical objects (ibid.) and there is no sign that he ever questioned his view that truth and falsity are logical objects.[18]

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17 Frege, as has been often noted, had reservations about the fifth axiom of BL right from the beginning, even before learning of the contradiction.

18 In IF (1981) p. 464 Dummett writes that for the late Frege (in "Sources of Knowledge..." of 1924/5) "there are no objects given by logic alone". I don't think this is correct. Frege indeed denies there that sets are

It may be instructive to consider a passage Frege wrote to Russell (after learning of the contradiction) on 28.7.1902:

> "I myself was long reluctant to recognize ranges of values and hence classes; but I saw no other possibility of placing arithmetic on a logical foundation. But the question is, How do we apprehend logical objects? And I have found no other answer to it than this, We apprehend them as extensions of concepts, or more generally, as ranges of values of functions" (PMC 140-1).

Classes or courses of values are not the only logical objects in Frege's system. There are also two other important logical objects - the two truth-values - the True and the False. These two, as we shall see, are obtained by a similar move from equivalence to identity, although in this case the original move, in SR, is not made by Frege explicitly. In SR truth-values are introduced as the references of sentences by a somewhat strange argument to the effect that if we are concerned with the references of proper names (or sub-sentential expressions), this cannot be because of the thought expressed by the sentence (for which they are irrelevant), but only for its truth-value, which must then be regarded as the reference of the sentence (33/63). Frege then proceeds to say in a uniquely uncritical move:

> "Every declarative sentence concerned with the reference of its words is therefore to be regarded as a proper name, and its reference, if it has one, is either the True or the False. These two objects are recognized, if only implicitly, by everybody who judges something to be true..." (ibid.)

Even if we are ready to accept this strange version of the slingshot argument - ascribing a reference to a sentence - Frege's further move of regarding this reference as an object and the sentence as its proper name might still appear singularly strange and unmotivated. But the idea is quite similar to the one we have seen above with regard to the introduction of value-ranges as objects: an irresistible move from equivalence to identity, from 'P iff Q' to 'p=q', (the capitals stand for sentences, combined into a complex sentence by a sentential operator, and the small letters stand for the appropriate names of truth values). Putting the point in a way more similar to Frege's principle, we might introduce a special operator '*' carrying propositions (or thoughts or the contents of statements, whatever they are) into objects. The transition would then be from '$P \leftrightarrow Q$' to '$*P = *Q$'. We may accordingly formulate a principle, somewhat analogous to axiom V, which would on the one hand express, and on the other be justified by, this elementary logical transition: (*) $*P = *Q \leftrightarrow (P \leftrightarrow Q)$. Again, this irresistible

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objects, and speaks of the dangerous ways in which language can mislead us to postulate objects where there are none. He even speculates that number-words are not names and do not designate objects. But he does not say that there are no logical objects.

move from equivalence to identity expresses a feature of the way we get to these objects - logical objects that are given to us by way of logic.[19]

Here, within the domain of truth-functional logic the equivalence plays a somewhat similar role to that of the general equivalence in the predicate logic. Both express individuation conditions of the basic units of the logical domains concerned: truth and falsity here, functions and their range-values there. In both cases Frege assumed (once explicitly, once implicitly) an elementary and irresistible transition from the equivalence in question to identity between objects. This is the crucial step in which he introduced (or discovered) logical objects as the basis of the objectivity of logic.[20] These objects are the basis of the objectivity of logic in the sense that their modes of presentation, expressed by the elementary transitions involved, justify the basic truths of the said logical domains: "The truth of a logical law is immediately evident from itself, from the sense of its expression" (CP 405). This, according to Frege, is the only way in which such basic truths (axioms) can be justified, hence they form the basis of their objectivity. Senses, according to the core idea, are such modes of presentation, which means that they stand at the very basis of the objectivity of the logical axioms.[21]

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[19] In my article, "Identity and the Formation of the Notion of Object", Erkenntnis 17 (1982), pp. 229-48, I proposed that the ontology of a theory is determined by the "identity stipulation", which identifies a general equivalence relation (indiscernibility defined over all the descriptive predicates of the language) with identity. I was not aware then of the affinity between this idea and the basic Fregean idea I am trying to present here.

[20] Frege calls axioms and basic logical truths "general" (e.g. FA section 3). This means, I suppose, that no particular objects (or functions) are referred to. It seems to me that this does not conflict with my emphasizing objects and their ways of being given as an epistemic ground of the axioms, for I am obviously talking about kinds of objects (truth-values, courses of values of functions) and not particular objects. For example, the propositional axiom $a\supset(b\supset\supset a)$ is, in Frege's mature conception, a generalization over truth-values. It does not refer to a particular truth-value or object. On the conception I propose here it expresses a feature of the ways truth-values are given to us, wherein its justification lies.

[21] The above formulations are quite general and should be qualified in various ways. One of the most important is this: It could be argued that the logical transition mentioned does not express the sense of, e.g., truth-values, for the sense of a truth-value is a particular thought, and the sense of, e.g., a class is certainly more complex than the said elementary logical transition, and contains an element that relates to the sense of the function determining that class. This is true when the sense of a particular truth-value or a particular class is concerned. I am discussing more generally the way in which truth-values and classes are given to us. A thought expresses the sense which "belongs" (as Frege says) to its truth-value, but this is subordinate to the mode in which truth-values are given to us as objects in general. Similarly, a particular sense belongs to a class under the mode in which classes are given to us in general. The elementary transitions discussed are intended to express such general features of the mode of presentation of logical objects.

(D) Frege's Principle and Equivalence-Relations

The equivalence involved in Frege's fundamental principle is not exactly an equivalence-relation, but rather a propositional equivalence. Not just any definition in terms of equivalence relation discovers or presents objects in the sense in which logical objects are presented by Frege's principle. One might fix the meaning of identity between such things as spatial directions or income levels in terms of the appropriate equivalence relations. But these would not amount to "discovering" these objects and would not express the ways these "objects" are given to us, similarly to the way I described above with regard to logical objects. Some light may be shed on this if we compare the above with what might seem to be a similar move in FA. It must of course be remembered that in FA Frege held neither the systematic distinction between sense and reference, nor the doctrine of truth-values as the references of sentences. Moreover, the correlative doctrine of sentences as proper names of objects (the truth-values) seems to conflict with the emphatic conception in FA of sentences as being radically different from names, and of their having a unique and primary character.

In a famous section Frege prefaces to his proposed definition of number in FA (#64) he says that we can introduce directions (as objects) by stipulating that the directions of two lines are identical iff the lines are parallel ($D(a)=D(b) \leftrightarrow a//b$). This again is a sort of transition from an equivalence to strict identity. Frege agrees with Kant that lines and parallelism are grounded in our (spatial) intuition, and the above principle expresses (at least an important aspect of) a way in which directions might be given to us as objects.

There is, however, a great difference between this example and the general point we made above. The equivalence we talked about in connection with axiom V and (*) is "propositional equivalence"; from a modern point of view, it is, on the face of it, not a relation at all, but is expressed by a truth-functional connective between sentences. The equivalence on the basis of which Frege introduces directions in FA, in contrast, is an ordinary equivalence-relation, which is a relation between objects.[22] The difference I have in mind may seem somewhat non-Fregean in spirit. For in *Begriffsschrift* Frege introduces "identity of content" for "judicable" (i.e. propositional) and non-judicable contents alike. But, besides a notorious confusion of use and mention in that early discussion, the distinction concerned (between propositions and objects) became to be of central importance for Frege in FA. The difference is again blurred by Frege's later view that equivalence (like other propositional

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[22] The status of the general equivalence relation of "indescernibility" in the "identity stipulation" in my paper in note 18 above is more complicated. It may appear that it is an ordinary equivalence relation between objects. This however, is not exactly so. The indiscernibility relation is intentionally defined in terms of propositional equivalence with no presumption of an objectual construal of the semantics involved. The whole point of the "identity stipulation" proposed there is to fix the ontology of the language on the basis of a non-objectual basis.

operations) is a (first-order) relation of truth-values. I think, though, that the difference holds also for this late view, since even on this view, truth-values are very special objects, and there are reasons to believe that in his mature and late writings Frege continued to adhere to his context principle and the primacy of propositions proclaimed in FA. But I shall not go into this much debated point, for here I am concerned with FA, in which Frege had not yet held this view of truth-values as objects.[23]

The definition of directions in terms of the equivalence relation of parallelism, discussed in FA, is, therefore, not an instance of that transition from (propositional) equivalence to identity of which we spoke above. The problem with such definitions is not only that they are contextual - that what is defined is only the whole identity context of, e.g., D(a)=D(b). It is also the (related) problem that they are definitional stipulations that fix the mode of presentation "vacuously", and do not express an independently existing mode of presentation of the objects concerned. The definition as a whole is therefore not a justifiable truth (though, as Frege himself remarked, it itself may justify a corresponding truth). This is part of what Frege means by saying that one can understand that a//b with no appeal to directions at all. With regard to truth-values, on the other hand, Frege asserts that "these objects are recognized, if only implicitly, by everybody who judges something to be true". The move from parallelism to identity of directions is not the sort of irresistible, fundamental transition in terms of which Frege describes his axiom V.

Directions are not (basic) objects on which the objectivity of geometry rests. Accounting for the way in which they are given to us is thus no part of an account of the objectivity of geometry. Directions <u>can</u> be introduced into geometry by the above principle. They <u>can</u> be also explicitly defined, on familiar lines, on the basis of the principle. But this is not necessary for the objectivity of geometry. What is required is a conception of the basic geometrical objects - lines and points. These are given to us in a particular way expressed (in part) by the geometrical axioms, thus forming a justification for the axioms themselves.[24] The

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[23] In a letter to Russell of 28.7.1902 Frege, mentioning his view in FA, explains the transition from equivalence-relation to identity as the general principle behind Russell's "definition by abstraction". He then remarks that the difficulties involved with this procedure of definition by abstraction are the same as those of "transforming the generality of an identity into an identity of ranges of values" (PMC, p. 141). The English translation here mistakenly put "not" into the sentence (probably a misprint), as if the difficulties concerned in the two cases are different. The original is: "Die Schwierigkeiten sind hierbei aber dieselben, wie bei der Umsetzung der Allgemeinheit einer Gleichheit in eine Werthverlaufsgleichheit" (p.224). I thank the editors for pointing this out to me.

[24] Cf. Boolos (1990) p. 248. Boolos argues that the Fregean analogy between "the direction of l" and "the number of Fs" is misleading, because "we do not suspect that lines are made up of directions, that directions are some of the ingredients of lines". This is another way of saying what I argue in the text - that directions are not basic geometrical objects. But in contrast to Boolos, I believe that this substantiates Frege's

logical case we discussed above is similar: Here we are looking for the basic objects whose modes of being given to us form the basis of the objectivity of the logical domain. What, then, is the point of the example about directions, which Frege discusses at such length?

In FA Frege did not pretend to have "discovered" that there are arithmetical objects, nor was this his aim. That numbers are objects is a fact he was convinced of prior to the logical analysis of their nature. If numbers in arithmetic were the analogues of lines and points in geometry, Frege would have ended up with a sort of Kantian conception of arithmetic, believing that the arithmetical axioms are sui generis and irreducible (like the axioms of geometry). But that, of course, was not his view. The task of defining the natural numbers (and the concept of a natural number) was designed to prevent this result. It was designed to show that the axioms are not sui generis in this manner. Hence, the numbers are the analogues of directions in geometry rather than of lines. As directions can be defined in terms of "basic" geometrical objects - lines, objects whose mode of being given justifies the axioms - so numbers can be defined in terms of "basic" logical objects - courses of values of functions, objects whose mode of being given to us justifies the basic truths of the domain.

(E) Digging into the Self-Evident

In order to sharpen the edges here, I will compare some aspects of the above account with some recent alternatives. Fundamental logical principles (rules of inference as well as axioms) were considered self-evident by Frege. And this has been presented by many authors as their ultimate justification. There is no going beyond this point, according to this conception; this is the end point of the justification-game. Some people regard any attempt to step "beyond" this point as betraying a misunderstanding of Frege's conception of the "autonomy" of logic, the view that there is no "meta-logical perspective" (to use Ricketts' suggestive expression; see his 1986) from which the fundamental logical truths may be justified. It may seem that our previous account about the role of sense in justifying the basic truths of a domain (including logic) is opposed to that, as if we have suggested an epistemological perspective (grasping the ways in which logical objects are given to us) precisely as such meta-logical perspective.

This may indeed be the case, if logic is narrowly construed as being concerned solely with deductions and proofs. On this narrow conception any attempt to view logical axioms as within the justification space, as being susceptible to any form of justification would amount to attempting a meta-logical perspective. I have argued, however, that Frege's conception of logic was broader: logic, on this broad conception, inheres in, and actually constitutes, the entire justification-space; it is concerned with whatever is relevant to justification. If this broad use of "logic" is adopted (in line, I believe, with Frege, as well as many other

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analogy, as the point of these sections in FA is to show that numbers are not "basic" objects either - they are definable in terms of the logical objects. Hence the analogy does serve a point.

nineteenth-century logicians), and logic is construed as the science of justification and objectivity, then the theses of the autonomy of logic and the lack of meta-logical perspective are correct. But then the sense-oriented justification of the basic logical principles and an account of their objectivity are within the logical enterprise. It is not a step beyond logic.

Tyler Burge has rightly emphasized Frege's epistemological concerns in this connection (see Burge 1992, pp. 645-649). However, he characterizes the self-evidence of the basic logical truths in terms of understanding: "[Frege] did see [the basic logical truths] as a source for the justification of the belief in them by a *person who understood them*" (645, my emphasis - G.B). Burge then (rightly again) connects "understanding logical truth" with "understanding the nature of justification for our mathematical judgments" (646) and tries to explain this in terms of Frege's late conception of logic, in *Der Gedanke,* as the science of the laws of truth ("what is") and as the normative science of the laws of thought. The connection is couched in very general terms - the general notions of reason and understanding - and as such seems to me of limited explanatory value. In any case the line I suggest in the text is much more specific in its reliance on a fully fledged notion of sense as a mode of presentation, and on the two faces of objectivity - justification, on the one hand, and relation to objects, on the other.

Burge expands on these themes in his more recent paper of 1998. Frege proposed ways of justifying the basic truths of his logic in what Burge calls "justification through application" (Burge 1998, pp. 330-335). He rightly comments that these applications serve as partial, inductive "tests" of the logical system (1998, p. 332). With regard to the axioms he says that "the recognition of advantages seems to provide a prima facie, probabilistic justification [...] Such recognition may provide indirect grounds for believing that the axioms are indeed basic and indeed true. But the supposed self-evidence of the axioms is ideally the primary source of their justification" (334). The self-evidence of the axioms is not a subjective, or even an epistemical, conviction. It is an objective property concerning which we are fallible, as we may wrongly judge a proposition to be self-evident. We may be wrong either in ascribing self-evidence to a proposition that lacks it, or, inversely, in failing to recognize it where it inheres. And if a proposition is self-evident we come to recognize this by a thorough understanding of it (339).

Understanding a truth may come in various degrees and may involve various factors, including, e.g., the inferential relations it has within a system. But what I am suggesting here is that a thorough understanding of a basic truth must also include a feature of its justification (or grounding), which, in the case of a truth, consists in its expressing the ways in which its objects present themselves to us - their sense. Frege says that "the truth of a logical law is immediately evident from itself, from the sense of its expression" (CP 405). We should take "sense" here very seriously. Understanding a law is grasping its sense, the thought expressed by it; and a thorough understanding is a thorough grasping of the sense, which includes knowing the way it is built up by its constituent senses. These, in typical cases, include the modes of presentation of the objects of these thoughts. Thus recognizing the truth of an axiom and justifying a basic truth are grounded in grasping the senses of their constituents, the modes of presentation of their objects.

Burge points out that although, in some sense, Frege presents arguments for his axioms (e.g. in section 18 of BL), these arguments are neither semantic, nor justificatory. They are not semantic in that they don't mention symbols, and they are not couched in a semantic meta-language. They are not justificatory in that they do not provide a justification of the axiom on the basis of other, more basic axioms. How then should these explanations or arguments to be regarded?

They should be regarded, Burge claims, as explications or articulations of the meaning of the axioms, as expressing a proper understanding of them (1998, pp. 316-329). I basically agree with all that, but wish to emphasize a point that seems to me crucial. With regard to his first axiom, $a\supset(b\supset\supset a)$, Frege argues, or explains, that "it could be false only if both a and b were the True while a were not the True. This is impossible. Therefore $a\supset(b\supset\supset a)$" (BL, section 18). Comparing this to its "counterpart" passage in *Begriffsschrift*, Burge concludes that "The argument serves to articulate understanding of the thought content. It does so in a way that enables one to recognize that its truth is guaranteed by its content" (1998, p.317). This may be an apt description of the *Begriffsschrift* passage, but the BL one contains more. On the face of it, there does seem to be an argument there: Frege articulates what would be required for the axiom to be false, in a way that makes it manifest that this is impossible, and concludes that it must be true. Burge may be right that this is not a semantic argument, and that it does not prove the axiom on the basis of other truths. What then is the point of the argument? The argument, I suggest, couched in the object language, highlights features of the ways the objects concerned (the two truth-values) are given to us (ways constrained by the principles of the classical truth tables presumed here) as justifying the axiom, and laying its truth open to view. Frege does not merely appeal to the meaning of the axiom and to our understanding of it; he appeals to it in a particular way that makes manifest the way its objects are presented to us by appealing to the principles constraining this way.

The justification relation between senses and basic truths is not a foundational one. It is not that senses are prior to or more basic than logical principles. The principles express aspects of the senses of the fundamental objects of the domain, whereby they (or our beliefs in them) are justified.[25]

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[25] P. Simons pointed out that Frege's note 16 to section 10 of BL, when read in light of FC, suggests that "the two sides of [Axiom] V express the same sense but in different ways" (Simons, 1992, p.765) The fact that Frege could think so fits the non-foundational character of his conception of justification alluded to in the text.

Justification is not construed here as what one might call a purely epistemic notion: It is not concerned only with the question of how a belief in the truth of a certain principle is justified, but also with the (ontological) question of what such a truth amounts to. On this Frege held a firmly realistic view according to which such truths are about objects and their properties and relations, and this, for him, was an essential feature of their truth. The justification I am talking about concerns modes of presentation of these objects and properties - their senses - in the fully fledged senses of these notions. I emphasize this in order to distinguish this position from others such as Peacocke's (1992), which seem to avoid such realistic commitments to objects and their modes of presentation. Peacocke presents a theory that connects what he regards as a Fregean notion of sense with justification. But, leaving aside for the moment the connection itself, it seems to me that the notion of sense alluded to is not the fully fledged Fregean notion of mode of presentation. What Peacocke presents is in fact a version of a Davidsonian conception which in this respect is clearly non-Fregean.

Peacocke's idea is that a system of "understanding conditions" or "possession conditions" for a certain expression (conditions accepted by anyone who understands the expression) determine a "semantic value" for that expression. In fact Peacocke assumes that, under certain requirements, it determines the semantic value uniquely. The semantic value may then justify principles including the expression, in the sense that, given the semantic values of the expressions involved, the principle may be shown to be correct (803; section 4). Now, Peacocke realizes that such semantic values are not Fregean senses - they have different modes of presentation and may be given to us in various ways. Peacocke assumes, however, that the main effect of Fregean sense can be accomplished by a theory of "canonical derivation", according to which, e.g., the truth conditions of a sentence may be "canonically derived" from the semantic values of its constituent expressions. But such a theory, if can be worked out, would not actually be using Fregean senses. On the contrary, it is designed to bypass them, to show them to be dispensable. For Frege, in contrast, Fregean senses — modes of presentation of objects and functions - are indispensable in an account of the objectivity and justification of the basic truths of any domain, including logic.

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